Reducible Graphs and Bisimilarity of 1-free Star Expressions

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A process interpretation of regular expressions was first studied in Milner (1984), where they were called *star expressions*. Milner gave a sound axiomatization for bisimilarity of the process interpretation of star expressions, leaving the problem of completeness open. Recently, Grabmayer & Fokkink (2020) gave a partial solution for 1-free star expressions via *LLEE charts*. We propose a new class of charts, based on *reducible graphs*, an important class of graphs in compiler engineering. These graphs can be characterized via a structural property and via graph transformations; we leverage both to construct a simple completeness proof for *strongly normed* 1-free star expressions. We hope to extend this work in progress to a full completeness proof, and are interested in other applications of structural characterizations in term graph rewriting.

1 Star Expressions and Process Semantics

In 1984, Milner studied *star expressions*, i.e., regular expressions up to bisimilarity [12]. He showed that all but two of Salomaa's axioms for regular language equivalence [15] are *sound* in this setting, and asked whether this subset might be *complete*. In 2020, Grabmayer and Fokkink [6] made substantial progress on this question by settling its restriction to 1-*free star expressions*. These are regular expressions without the constant 1, and where the Kleene star takes its (original [10]) binary form:

$$\mathsf{StExp}(A) \ni e, f ::= 0 \mid a \in A \mid e + f \mid e \cdot f \mid e^{\circledast} f$$

To fully appreciate this result, we need to develop some formal notions first, starting with charts.

Definition 1.1. We fix a finite set of actions A. A *chart* is a tuple $C = (V, T, v^s)$ where V is a finite set of *nodes*, $T \subseteq V \times A \times (V \cup \{\surd\})$ is a *transition relation* where $\surd \notin V$ is a special *terminating* node, and $v^s \in V$ is the *initial node*. We write $v \xrightarrow{a} v'$ to mean $(v, a, v') \in T$. A node $v' \in V \cup \{\surd\}$ is said to be *reachable* from a node $v \in V$ if there exist $v = v_0, \ldots, v_n = v'$ such that for $0 \leq i < n$, there exists an $a \in A$ with $v_i \xrightarrow{a} v_{i+1}$. We call $v \in V$ *normed* if $\surd$ is reachable from v. We assume that all charts are *start-vertex-connected*, meaning that all vertices are reachable from v^s.

Importantly for this work, we call a chart *strongly normed* if every node in the chart is normed. We can define a transition relation on 1-free star expressions, and in turn a chart for every 1-free star expression, called its *chart interpretation*.

Definition 1.2. The transition relation $T(A)$ is the smallest subset of $\mathsf{StExp}(A) \times A \times (\mathsf{StExp}(A) \cup \{\surd\})$ that satisfies the following rules, for all $e_1, e_2, e_1' \in \mathsf{StExp}(A)$, $a \in A$ and $\xi \in \mathsf{StExp}(A) \cup \{\surd\}$:

$$\frac{}{a \xrightarrow{a} \surd} \qquad \frac{e_1 \xrightarrow{a} \xi}{e_1 + e_2 \xrightarrow{a} \xi} \qquad \frac{e_2 \xrightarrow{a} \xi}{e_1 + e_2 \xrightarrow{a} \xi} \qquad \frac{e_1 \xrightarrow{a} e_1'}{e_1 \cdot e_2 \xrightarrow{a} e_1' \cdot e_2} \qquad \frac{e_1 \xrightarrow{a} \surd}{e_1 \cdot e_2 \xrightarrow{a} e_2}$$

$$\frac{e_1 \xrightarrow{a} e_1'}{e_1^{\circledast} e_2 \xrightarrow{a} e_1' \cdot (e_1^{\circledast} e_2)} \qquad \frac{e_1 \xrightarrow{a} \surd}{e_1^{\circledast} e_2 \xrightarrow{a} e_1^{\circledast} e_2} \qquad \frac{e_2 \xrightarrow{a} \xi}{e_1^{\circledast} e_2 \xrightarrow{a} \xi}$$

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For every $e \in \mathsf{StExp}(A)$, this transition relation gives rise to a chart $C(e) = (V(e), T(e), e)$, where $T(e)$ is generated by the transitions starting from e, and $V(e) \subseteq \mathsf{StExp}(A)$ is the finite set of star expressions occurring in $T(e)$. We call an expression e *normed* if it is normed in $C(e)$, and *strongly normed* if $C(e)$ is strongly normed. Intuitively, an expression e is strongly normed when it always terminates successfully.

Charts can be equipped with a notion of bisimulation and homomorphism, as follows:[1]

Definition 1.3. Given two charts $C_i = (V_i, T_i, v_i^s)$ for $i \in \{1,2\}$, a *bisimulation* between C_1 and C_2 is a relation $R \subseteq V_1 \times V_2$ such that $v_1^s R v_2^s$, and if $v_1 R v_2$, the following conditions hold:

(*forth*) If $v_1 \xrightarrow{a} w_1 \in V_1$ in C_1, then there exists a transition $v_2 \xrightarrow{a} w_2$ in C_2 such that $w_1 R w_2$;

(*back*) If $v_2 \xrightarrow{a} w_2 \in V_2$ in C_2, then there exists a transition $v_1 \xrightarrow{a} w_1$ in C_1 such that $w_1 R w_2$;

(*termination*) $v_1 = \surd$ if and only if $v_2 = \surd$.

If there is a bisimulation between C_1 and C_2, we say that they are *bisimilar* and denote this by $C_1 \leftrightarrows C_2$. A function $f : V_1 \to V_2$ is a *homomorphism* of charts if the graph of f is a bisimulation. Finally, we call $e, f \in \mathsf{StExp}(A)$ *bisimilar*, denoted $e \leftrightarrows f$, when their charts are bisimilar, i.e., when $C(e) \leftrightarrows C(f)$.

The proof system BBP for 1-free star expressions up to bisimilarity is given as follows:

$$(B1)\ x+y = y+x \qquad (B2)\ (x+y)+z = x+(y+z) \qquad (B3)\ x+x = x \qquad (B4)\ (x+y)\cdot z = x\cdot z + y\cdot z$$

$$(B5)\ (x\cdot y)\cdot z = x\cdot(y\cdot z) \qquad (B6)\ x+0 = x \qquad (B7)\ 0\cdot x = 0 \qquad (BKS1)\ x\cdot(x^{\circledast} y)+y = x^{\circledast} y$$

$$(BKS2)\ (x^{\circledast} y)\cdot z = x^{\circledast}(y\cdot z) \qquad (RSP)\ x = (y\cdot x)+z \implies x = y^{\circledast} z$$

Notice how, in contrast with Salomaa's axioms for language equivalence of regular expressions, the left-distributivity axiom ($x\cdot(y+z) = x\cdot y + x\cdot z$) and the right-annihilation axiom ($x\cdot 0 = 0$) are absent.

We write $e_1 =_{\mathrm{BBP}} e_2$ to mean that $e_1 = e_2$ is derivable from the axioms above, together with the laws of equational logic. Milner showed that BBP is *sound* for bisimilarity, that is, if $e_1 =_{\mathrm{BBP}} e_2$, then $e_1 \leftrightarrows e_2$. The converse, known as *completeness*, is the result proved by Grabmayer and Fokkink [6].

Theorem 1.4. *BBP is complete w.r.t. bisimilarity: for any $e_1, e_2 \in \mathsf{StExp}(A)$, $e_1 \leftrightarrows e_2$ implies $e_1 =_{BBP} e_2$.*

The main technical hurdle to prove this theorem is that some charts C are not bisimilar to any 1-free star expression, i.e., $C \leftrightarrows C(e)$ does not hold for any 1-free star expression e. Indeed, Milner had already noticed this [12] (for star expressions at large), and asked for a characterization of the charts that *can* be described by expressions. Grabmayer and Fokkink proposed a property of charts called *layered loop existence and elimination* (LLEE), and show (1) that the chart of any expression is LLEE, and (2) an equivalent expression can be recovered from any LLEE chart. Additional properties of LLEE charts (see the next section) can then be leveraged to prove Theorem 1.4, in a manner reminiscent of the way Kleene's theorem [10] can be used to prove completeness of Kleene algebra for regular expressions [11].

Contribution We show that the proof strategy used by Grabmayer and Fokkink can be used to achieve a completeness result (currently limited to strongly normed expressions) if LLEE charts are replaced with a different class of charts that we call *reducible* charts. What makes this interesting is that reducible charts admit a description that is arguably simpler than LLEE charts. Moreover, reducible charts correspond

[1] Every chart $C = (V, T, v^s)$ induces functions $o : V \to 2^A$ and $\delta : V \to P_f(V)^A$, where $o(v)(a) = 1$ iff $v \xrightarrow{a} \surd$, and $\delta(v)(\alpha) \ni v'$ iff $v \xrightarrow{a} v'$. Thus, a chart $C = (V, T, v^s)$ is a pointed coalgebra for the functor $2^A \times P_f(-)^A$. The notions of bisimilarity and homomorphism correspond precisely to bisimulation and homomorphism as derived from this representation; cf. [14].

closely to (inverted) *reducible flow graphs* [1], a class of graphs that is important in compiler engineering; in fact, theoretical results about reducible flow graphs [8] form the basis of our development, which in turn significantly simplify the proof of one crucial property (closure under homomorphisms).

The remainder of this paper is organized as follows. Section 2 gives a general approach to completeness proofs for (1-free) star expressions, along with the properties that LLEE and reducible charts must satisfy. Section 3 introduces reducible graphs. Section 4 shows that reducible charts satisfy all properties necessary for the limited completeness proof. Section 5 concludes with some directions for future work.

2 A General Approach to Completeness Proofs

We now go over the proof strategy for completeness. In general terms, this strategy is very similar to the one described in [6], although our development will follow what is called the "global approach" in [17]. The first formal notion that we will need is that of a *solution* to a chart.

Definition 2.1. A *(provable) solution* to a chart $C = (V, T, v^s)$, is a function $s : V \to \mathsf{StExp}(A)$ such that the following holds for all $v \in V$, where the sums are the obvious generalizations of $+$:

$$s(v) =_{\text{BBP}} \sum_{v \xrightarrow{a} \surd} a + \sum_{v \xrightarrow{b} w} b \cdot s(w)$$

Importantly, solutions to chart interpretations of star expressions are given by the expressions themselves. Formally, the identity function $\mathsf{id} : V(e) \to V(e)$ is a solution to the chart $C(e)$ for every star expression e. The crux of the completeness proof is to find a class of charts $\mathscr{C}$ satisfying three conditions:

(I) The chart interpretation of every star expression is in $\mathscr{C}$;

(II) Every chart in $\mathscr{C}$ admits a solution, unique up to provable equivalence; and

(III) The class $\mathscr{C}$ is closed under homomorphic images.

Briefly, given these three properties, one can prove completeness as follows. First, let R be the bisimulation between $C(e_1)$ and $C(e_2)$ witnessing that $e_1 \leftrightarrows e_2$. Using standard techniques, one can then construct a chart $C = (V, T, v^s)$ whose nodes are equivalence classes of $V(e_1) \cup V(e_2)$ under R, and show that the quotient maps q_i from $V(e_i)$ to V are surjective homomorphisms. Because $C(e_1)$ and $C(e_2)$ are in $\mathscr{C}$ by (I), so is C, by (III). Now let s_i be the unique solution to $C(e_i)$, and let s be the unique solution to C, which exist by (II). A straightforward argument (independent of $\mathscr{C}$) shows that $s \circ q_i$ is a solution to $C(e_i)$. Uniqueness of solutions (II) then implies that $e_1 =_{\text{BBP}} s_1(e_1) =_{\text{BBP}} s(v^s) =_{\text{BBP}} s_2(e_2) =_{\text{BBP}} e_2$.

It then remains to propose a class $\mathscr{C}$ and prove that properties (I) through (III) hold. To this end, the class of *LLEE charts* is developed and shown to be suitable in [6]. We now recall an equivalent but somewhat more compact description of this class, known as *well-layered charts* [17].

Informally, a chart $C = (V, T, v^s)$ is well-layered if its loops are not mutually nested; this property is recorded in a *layering witness*. Because the layering witness makes no reference to the actions or the initial node, it can be defined on the underlying transition graph $C^\bullet = (V_\surd, T^\bullet)$, where $V_\surd = V \cup \{\surd\}$ and $T^\bullet : V_\surd \to P_f(V_\surd)$ is given by $T^\bullet(v) = \{w \in V \mid \exists a \in A.\, (v, a, w) \in T\}$. An *entry/body labelling* of such a transition graph is a labelling of each transition $x \to y$ as either an *entry* transition $x \to_e y$, or a *body* transition $x \to_b y$. We use the notation $x \curvearrowright y$ to mean $\exists(v_1, \ldots, v_k).x \to_e v_1 \to_b \ldots \to_b v_k \to_b y$.

Definition 2.2. A *layering witness* for a transition graph $(V_\surd, T^\bullet)$ is an entry/body labelling that is:

(fully specified) for all $v, w \in V_\surd$, (a) $\neg(v \to_b^+ v)$ and (b) if $v \to_e w$ and $w \neq v$, then $w \to^+ v$;

(layered) the directed graph $(V_{\surd}, \curvearrowright)$ is acyclic; and

(goto-free) for all $v, w \in V_{\surd}$, $v \curvearrowright w$ implies $\neg(w \to \surd)$.

A chart (V, T, v^s) is *well-layered* if its underlying transition graph $(V_{\surd}, T^\bullet)$ admits a layering witness.

Given that in [6] properties (I) through (III) were effectively already shown for LLEE charts[2] , they also hold for well-layered charts, although [17] also contains a proof specialized to well-layered charts. Crucially, in both of these works, the proof of (III) is the most ingenous and as a result most complicated of the three, spanning several pages.

3 Reducible Graphs

The control flow of sequential programs is often modelled by a directed graph called a *flow graph*, which allows for program analysis and transformation. Often, such analyses require a property called *reducibility*, first studied by Allen [1] in the 70s. In [8], Hecht and Ullman give two characterizations of reducible flow graphs, one via graph transformations and one via a structural property. In this section, we introduce reducible graphs and adapt the relevant results to our setting. By dualizing reducibility we obtain a class of graphs that, as we will see in the next section, satisfies properties (I) through (III).

We first recall some standard graph-theoretic notions. A finite *directed graph* $G = (V, E)$ is a pair where V is a finite set of vertices and $E \subseteq V \times V$ is the edge relation. A graph $G' = (V', E')$ is said to be a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. The *converse* of $G = (V, E)$ is $G^{-1} = (V, E^{-1})$, i.e., the graph G with the edges reversed. We denote the graph consisting of a single node with no edges by **0**.

Given a graph $G = (V, E)$, the edge $(n, m) \in E$ is said to *leave* node n and *enter* node m. In such a case, we call n a *predecessor* of m, and m a *successor* of n. A sequence of nodes $(n_0, n_1, \ldots, n_k)$, is a *path of length* k from node n_0 to node n_k if there is an edge (n_{i-1}, n_i) for $1 \leq i \leq k$. A *cycle* is a path $(n_0, n_1, \ldots, n_k)$ in which $n_0 = n_k$. If $k = 1$, then the cycle is a *loop*. If (n, n) is an edge, we say node *n has a loop*. We will denote an edge (n, m) in a graph by $n \to m$ when the graph is understood.

A graph is *rooted* if there exists at least one node r such that there is a path to all nodes from r. The node r is called a *root* of the graph. A *flow graph* (G, r) is a graph together with a distinguished root r, which is called the *initial node*. We will be interested in the dual of the class of flow graphs that we call *coflow graphs*: graphs (G, t) whose converse (G^{-1}, t) is a flow graph. In this case, there exists a path from every node to t in G, and we call t the *terminal node* of G. Note that the terminal node is distinct from a sink node, which is a node with only incoming edges and no outgoing edges. Since an initial node in a flow graph may have incoming edges, a terminal node may have outgoing edges.

Hecht and Ullman [8] characterized reducible flow graphs as those graphs that can be transformed to the graph **0** via two graph transformations. We give the dual transformations here for coflow graphs.

Definition 3.1. Let $G = (V, E, t)$ be an coflow graph. Given a vertex $v \in V$ for which there exists a loop $(v, v) \in E$, the transformation $T_1(v)$ removes this loop, resulting in $G' = (V, E \setminus \{(v, v)\})$. Separately, given two vertices $v_1, v_2 \in V$ such that $v_1 \neq v_2$, $v_1 \neq t$ and v_2 is the unique successor of v_1, the transformation $T_2(v_1, v_2)$ merges v_1 into v_2, rerouting all edges entering v_1 to v_2. This results in $G' = (V', E')$, where

$$V' = V \setminus \{v_1\} \qquad E' = (E \cap (V' \times V')) \cup \{(w, v_2) \mid w \in V', (w, v_1) \in E\}$$

[2] To be more precise, in [6] it is shown that LLEE charts are closed under *bisimulation collapse*, which is slightly weaker than but closely related to closure under homomorphic images.

We define the relations $\Rightarrow_i$ for $i \in \{1,2\}$ on graphs, where $G \Rightarrow_i G'$ iff G can be transformed to G' by an application of T_i, and we let $\Rightarrow$ denote the union of $\Rightarrow_1$ and $\Rightarrow_2$. In [8], it was shown that this rewriting relation is locally confluent. Moreover, it is terminating, as each transformation step removes either an edge or a node. These two imply confluence by Newman's lemma [13]. Note that confluence technically only holds up to graph isomorphism. However, as will become will clear in the following section, this will not be a relevant issue for our work.

Lemma 3.2 ([8])**.** *The relation $\Rightarrow$ on coflow graphs is confluent: for all coflow graphs G_1, G_2 and G_3, if $G_1 \Rightarrow^* G_2$ and $G_1 \Rightarrow^* G_3$, then there exist coflow graphs H_1 and H_2 such that $H_1 \cong H_2$ and $G_2 \Rightarrow^* H_1$ and $G_3 \Rightarrow^* H_2$.*

We say that a coflow graph G is *reducible* if $G \Rightarrow^* \mathbf{0}$. Importantly, we can show that T_1 can be restricted to apply only to a node with at most one successor (besides itself), without affecting reducibility. We denote this restricted transformation by $T_1^\dagger$ and $\Rightarrow_{1\dagger}$.

Lemma 3.3. *For any coflow graph G, G is reducible iff $G \Rightarrow_\dagger^* \mathbf{0}$, where $\Rightarrow_\dagger$ is the union of $\Rightarrow_{1\dagger}$ and $\Rightarrow_2$.*

Proof. Clearly, if $G \Rightarrow_\dagger^* \mathbf{0}$, then G is reducible. Conversely, suppose that $\neg(G \Rightarrow_\dagger^* \mathbf{0})$; then let G' be such that $G \Rightarrow_\dagger^* G'$ and neither $T_1^\dagger$ nor T_2 can be applied to G'. It suffices to show that G' is not reducible, since $G \Rightarrow^* G'$ and $\Rightarrow$ is confluent. Since $T_1^\dagger$ and T_2 are not applicable to G', every node in G' must have at least two successors besides possibly itself. Let $\{v_1, \ldots, v_k\}$ be the nodes in G' with a loop, and let H be obtained from G' by applying T_1 to $\{v_1, \ldots, v_k\}$, so $G' \Rightarrow_1^* H$. Then neither T_1 nor T_2 are applicable to H, since all nodes in H must still have two distinct successors, and no node in H has a loop. Since $G \Rightarrow^* G' \Rightarrow^* H$ and H is not reducible, by confluence of $\Rightarrow$ we know G is not reducible. $\square$

Hecht and Ullman give a second structural characterization of reducible flow graphs as those not containing a specific subgraph shape. Its dual is the shape depicted in Figure 1, where t is the terminal node. Here, x, y, z and t are distinct, except that perhaps $z = t$. Furthermore, the squiggly edges between them denote pairwise *node-disjoint paths*, meaning that none of the paths share intermediate nodes. In graph-theoretic terms, such graphs are called *subdivisions* [19] of the graph where each path is a single edge. We call a coflow graph satisfying this shape *forbidden*.

Lemma 3.4 ([8])**.** *A coflow graph is reducible iff it does not contain any forbidden subgraphs.*

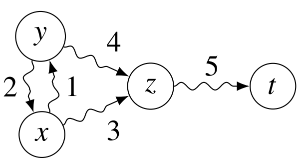


Figure 1: The forbidden graph schema

4 Reducible Charts for Star Expressions

A chart $C = (V, T, v^s)$ is *reducible* when its underlying coflow graph $(V_{\surd}, T^\bullet, \surd)$ is reducible. This implicitly requires that $\surd$ is the terminal node, which holds exactly for the class of strongly normed charts. The chart interpretations of star expressions that always terminate fall in this class, but expressions that allow non-termination, like $a^{\circledast}0$ or $a + (b \cdot 0)$, do not. We first provide a connection between LLEE-charts and reducible charts, as follows.

Lemma 4.1. *Every well-layered (LLEE) chart $C = (V, T, v^s)$ which is strongly normed is reducible.*

Proof. If $(V_{\surd}, T^{\bullet}, \surd)$ is not reducible, it contains a forbidden subgraph with nodes and paths labelled as in Figure 1, where the terminal node is $\surd$. Suppose there exists a layering witness for C. By iterating **(goto-free)**, it is clear that the transitions on the paths $5 : z \rightsquigarrow t$, $3 : x \rightsquigarrow z$ and $4 : y \rightsquigarrow z$ must all be labelled as body transitions. By **(fully specified)**, at least one transition on the paths $1 : x \rightsquigarrow y$ and $2 : y \rightsquigarrow x$ must be labelled as an entry transition, let this be $v \rightarrow_e v'$, which we can assume to be on path 1 without loss of generality. If no other transition on the cycle is labelled as an entry transition, we have a path $v \rightarrow_e v' \rightarrow_b^* y \rightarrow_b^* \surd$, contradicting **(goto-free)**. If there exists another entry transition $w \rightarrow_e w'$ on the cycle, then **(layered)** is violated. Since there can be no layering witness for C, it is not well-layered. □

The other direction remains open at present: are there (strongly normed) reducible charts that are not well-layered, or do the the two classes of charts coincide? We discuss this further in Section 5. For now, the previous lemma allows us to conclude that condition (I) applies partially, since every interpretation of a star expression is well-layered [6, 17]:

Lemma 4.2. *For every strongly normed $e \in \mathsf{StExp}(A)$, the interpretation $C(e)$ is reducible.*

We will derive solutions to reducible charts through their underlying coflow graphs. To this end, we generalize the labels on the edges of the underlying coflow graph of a chart to be 1-free star expressions instead of actions. This allows us to represent intermediate steps in the solution algorithm, while also ensuring that nodes are connected by a single edge. We call such a structure a *generalized chart*. Given a chart $C = (V, T, v^s)$, we label pairs of nodes $(v, w) \in V \times (V \cup \{\surd\})$ with expressions by summing the labels on transitions from v to w:

$$l_T(v, w) = \sum_{v \xrightarrow{a} w \in T} a$$

If there are no transitions from v to w, then $l_T(v, w) = 0$. Clearly, every chart (V, T, v^s) can be represented by its generalized chart (V, l_T, v^s); its underlying coflow graph connects $v \in V$ to $w \in V \cup \{\surd\}$ when $l_T(v, w) \neq_{\mathrm{BBP}} 0$.

We can define an analogous notion of a solution for generalized charts.

Definition 4.3. A *solution* to a generalized chart $C = (V, l, v^s)$ is a function $s : V \rightarrow \mathsf{StExp}(A)$ such that:

$$s(v) =_{\mathrm{BBP}} l(v, \surd) + \sum_{v \rightarrow w} l(v, w) \cdot s(w)$$

Any solution to (V, T, v^s) is also a solution to (V, l_T, v^s) and vice versa, by right distributivity. Thus, in order to solve a reducible chart, it suffices to construct a solution to its generalization. We will now argue that property (II) holds for generalized charts by applying transformations $\Rightarrow_{1\dagger}$ and $\Rightarrow_2$ directly to generalized charts (which affects their underlying coflow graphs in the analogous way) while relabelling transitions to preserve and reflect solutions. For the transformation $\Rightarrow_{1\dagger}$ applied to a node x, we relabel as depicted below.

$$\begin{tikzcd} x \arrow[loop above, "e"] \arrow[r, "f"] & y & \Rightarrow_{1\dagger} & x \arrow[r, "e^{\circledast} f"] & y \end{tikzcd}$$

It should be clear that if a generalized chart C_1 can be transformed to C_2 by $\Rightarrow_{1\dagger}$, then the same relation holds between their underlying coflow graphs, as those simply discard the labels on the edges.

Lemma 4.4. *Let $C_i = (V_i, l_i, v^s)$ for $i \in \{1,2\}$ be generalized charts such that $C_1 \Rightarrow_{1\dagger} C_2$. Then C_1 has a unique solution if and only if C_2 has a unique solution.*

Proof. Suppose that the transformation was applied to a node $x \in V$ with $l_1(x,x) = e$; then $l_2(x,x) = 0$ and $l_2(x,y) = e^{\circledast} l_1(x,y)$ where y is the unique successor to x such that $y \neq x$, with $l_1(v,w) = l_2(v,w)$ in all other cases. If s is a solution to C_1 then we can derive as follows, making use of the (RSP) axiom:

$$s(x) = e \cdot s(x) + l_1(x,y) \cdot s(y) = e^{\circledast}(l_1(x,y) \cdot s(y)) = (e^{\circledast} l_1(x,y)) \cdot s(y) = l_2(x,y) \cdot s(y).$$

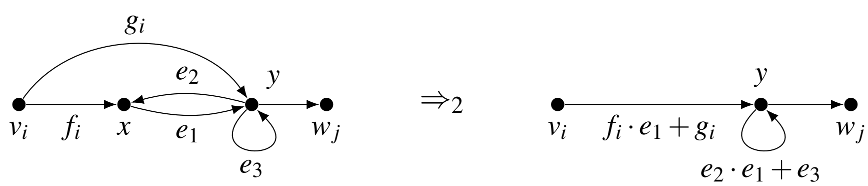
Thus, $s(x)$ is also a solution for x in C_2. Since no other structure changes in the graph, this means that s is a solution for C_2 as a whole. Conversely, suppose that s is a solution to C_2. Then we have the following:

$$\begin{aligned} s(x) &= (e^{\circledast} l_1(x,y)) \cdot s(y) \\ &= e^{\circledast}(l_1(x,y) \cdot s(y)) \\ &= e \cdot (e^{\circledast}(l_1(x,y) \cdot s(y))) + l_1(x,y) \cdot s(y) \\ &= e \cdot s(x) + l_1(x,y) \cdot s(y). \end{aligned}$$

Thus, $s(x)$ is a solution to x in C_1, and as the rest of the labels stay the same, s is a solution for C_1.

For uniqueness, suppose C_1 admits at most one solution, and let s, s' be solutions to C_2. Then s and s' are also solutions to C_1, and must be equal. The same argument works in the other direction. □

The transformation $\Rightarrow_2$ applied to a node x with a unique successor y merges the nodes, removing x and leaving only y. The relabelling of edges associated with this transformation is displayed below. Note that the edges labelled with e_2, e_3 and g_i need not exist, in which case the labels are simply 0.



Lemma 4.5. *Let $C_i = (V_i, l_i, v^s)$ for $i \in \{1,2\}$ be generalized charts such that $C_1 \Rightarrow_2 C_2$. Then C_1 has a unique solution if and only if C_2 has a unique solution.*

Proof. Suppose that T_2 was applied to a node $x \in V_1$ with unique successor y, where $l_1(x,y) = e_1$. Note that C_2 has the set of nodes $V_2 = V_1 \setminus \{x\}$. Given a solution $s_1 : V_1 \to \mathsf{StExp}(A)$ for C_1, we define a solution s_2 for C_2 by restricting the domain of s_1 to V_2. To verify that s_2 is a solution for G_2, it suffices to check that it is a solution for the nodes in C_2 whose outgoing transitions differ from C_1. This is only the case for y and every other node $v \in V_1$ where $v \to_1 x$, where we write $\to_i$ to mean the transition relation $T_i^{\bullet}$. This follows from a straight-forward verification in both cases.

Conversely, suppose that we have a solution $s_2 : V_2 \to \mathsf{StExp}(A)$ for C_2. Then we can define a solution $s_1 : V_1 \to \mathsf{StExp}(A)$ by extending s_2 as follows:

$$s_1(v) = \begin{cases} l_1(x,y) \cdot s_2(y) & \text{if } v = x \\ s_2(v) & \text{otherwise} \end{cases}$$

Since the outgoing transitions between C_2 and C_1 differ only for the nodes x and $v \in V_1$ such that $v \to_1 x$, it suffices to check that s_1 is a solution for these nodes, which is easily done. Finally, it is also easily verified that these transformations *uniquely* preserve and reflect solutions, by a similar argument as Lemma 4.4. □

Now that we have established that the transformations $T_1^{\dagger}$ and T_2 uniquely preserve and reflect solutions, we can prove that reducible charts are uniquely solvable.

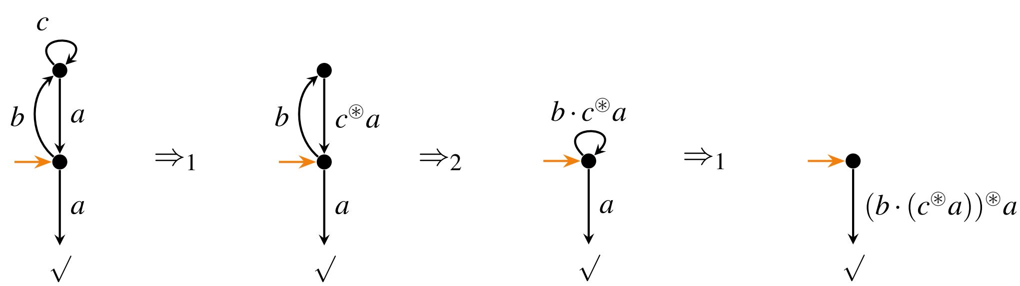
Lemma 4.6. *Every reducible chart admits a unique solution.*

Proof. Given a reducible chart (T,V,v^s), it suffices to construct a unique solution to its associated generalized chart $C = (V, l_T, v^s)$. Since C is reducible, we know that $C \Rightarrow_{\dagger}^{k} \mathbf{0}$ for some $k \in \mathbb{N}$ by Lemma 3.3. We prove the lemma by induction on k. Since we are interested in charts with nodes besides $\surd$, we can assume that the reducible chart is not already $\mathbf{0}$, so our base case is $k = 1$.

Case $k = 1$. We have $C \Rightarrow_{\dagger} \mathbf{0}$, which implies that $C \Rightarrow_2 \mathbf{0}$ since C contains some node besides $\surd$, which can only be removed by T_2. Thus, C is a graph with two nodes x and $\surd$, with a single labelled edge $x \xrightarrow{e} \surd$ for some $e \in \mathsf{StExp}(A)$. Clearly the unique solution to this graph is $s(x) = e$.

Case $k > 1$. We have that $C \Rightarrow_{\dagger} C' \Rightarrow_{\dagger}^{n} \mathbf{0}$ for some $n < k$. Thus, by our inductive hypothesis, C' admits a unique provable solution. Since $C \Rightarrow_{1\dagger} C'$ or $C \Rightarrow_2 C'$, we can apply Lemma 4.4 or Lemma 4.5 to obtain the fact that C admits a unique solution. ☐

Example 4.7. The transformations yield an algorithmic approach to constructing solutions to process charts. The following is an example of how a solution to a reducible chart is constructed via T_1 and T_2. The orange arrow points to the start state. Note that solutions to the other nodes can also be reconstructed, as described in the previous two lemmas. It is easily verified that the expression $(b \cdot (c^{\circledast} a))^{\circledast} a$ is indeed a (unique) solution to the starting state in the original chart.



The previous result gives us step (II) towards a completeness theorem. What remains now is step (III), that is, to show that the property of reducibility for charts is closed under homomorphic images. The relative simplicity of this proof compared to the analogous proof for LLEE charts [6, 17] makes reducibility an attractive notion to reason with. For the proof we use the structural characterization of reducibility.

Lemma 4.8. *Let $C_i = (V_i, T_i, v_i^s)$ for $i \in \{1,2\}$ be charts and let $f : C_1 \twoheadrightarrow C_2$ be a surjective homomorphism. If C_1 is reducible, then C_2 is too.*

Proof (sketch). The homomorphism f on charts is also a coalgebra homomorphism between their underlying transition graphs $C_1^{\bullet}$ and $C_2^{\bullet}$, which are coalgebras for the finite powerset functor. We will show that if $C_2^{\bullet}$ contains a forbidden subgraph, with nodes and labels as in Figure 1, then we can reconstruct a forbidden subgraph in $C_1^{\bullet}$ using surjectivity of f.

Let $a \in C_1^{\bullet}$ be such that $f(a) = x$. The path $1 : x \rightsquigarrow y$ in $C_2^{\bullet}$ is of the form $(x, x_1, \dots, x_k, y)$. This path is mirrored in $C_1^{\bullet}$ by f as a path $(a, a_1, \dots, a_k, b)$ where $f(a_i) = x_i$ for $i \leq k$, and $f(b) = y$. Similarly, the path $2 : y \rightsquigarrow x$ in $C_2^{\bullet}$ is of the form $(y, y_1, \dots, y_l, x)$, which is mirrored in $C_1^{\bullet}$ by f as a path $(b, b_1, \dots, b_l, a')$ where $f(b_i) = y_i$ for $i \leq l$, and $f(a') = y$. If $a = a'$, we have reconstructed the cycle in $C_1^{\bullet}$. If not, we can use a 'zigzag' argument, where we construct a new path from a' to some $b' \in f^{-1}(y)$ in $C_1^\bullet$ by surjectivity of f, and back from b' to some $a'' \in f^{-1}(y)$, and so on. Since there are finitely many nodes, the preimages $f^{-1}(x)$ and $f^{-1}(y)$ are finite. Thus, this construction must eventually repeat a node in one of these preimages, forming the cycle of the forbidden subgraph in $C_1^\bullet$. The remaining paths $3 : x \rightsquigarrow z$, $4 : y \rightsquigarrow z$ and $5 : z \rightsquigarrow t$ can be reconstructed in $C_1^\bullet$ similarly using f, taking care to ensure that the paths in $C_1^\bullet$ are disjoint. Thus, we can construct a forbidden subgraph in $C_1^\bullet$. □

5 Future Work

In summary, we have obtained the following results:

(I) For every *strongly normed* star expression e, its chart interpretation $C(e)$ is reducible.

(II) Every reducible chart has a unique provable solution.

(III) Reducible charts are closed under homomorphic images.

In total, this tells us that if e_1 and e_2 are strongly normed and $e_1 \leftrightarrows e_2$, then $e_1 =_{\mathrm{BBP}} e_2$. To recover Theorem 1.4 in full, we want to look at reducibility of graphs where not every node can reach the terminal node. We know that $\Rightarrow_{1^\dagger}$ and $\Rightarrow_2$ can still preserve and reflect solutions on such systems. However, we need to take into account that nodes may offer only transitions that never lead to acceptance; because these are not bisimilar to 0, they cannot be eliminated outright. We conjecture that reducibility in subcharts without termination can be characterized in a similar way. An inspiration for such a characterization is the complete directed graph on 3 nodes, which is another example of a chart that is not bisimilar to a star expression in [6].

If both of these turn out to be true, we would have a novel proof of Theorem 1.4. The obvious question is then whether the class of reducible charts coincides with LLEE charts, or whether they differ. Both cases would be interesting in their own right, and we plan to investigate this further. It would also be interesting to try and extend our results to bisimilarity for *all* star expressions. Recent work on completeness for all star expressions given by Grabmayer [5] builds on the LLEE property, but is rather intricate and complex as the LLEE property is no longer satisfied by every interpretation. A different approach to characterizing these process graphs may provide new insight into the problem.

Other avenues of further research include a coalgebraic generalization of our results to different types of behavior, such as (probabilistic) GKAT, in the style of [9]. We would also like to specialize our results to prove that normed expressions are bisimilar if and only if they are equivalent according to the laws of BBP extended with the right-absorption axiom $x \cdot 0 = 0$, similar to what was done in [16]. Furthermore, we wonder how the results are affected by extending 1-free star expressions with a specialized operator meant to model the exact behavior exemplified by the forbidden subchart, similar to what was done in [3].

Finally, we want to learn about other problems where graph transformations and structural characterizations could be applied, perhaps in the context of cyclic term graphs or nested term graphs [2, 7].

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