Relating ordered hypergraph rewriting and proof theory: Work in progress

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Proof Theory and Logic Programming: Computation as proof search, by Dale Miller

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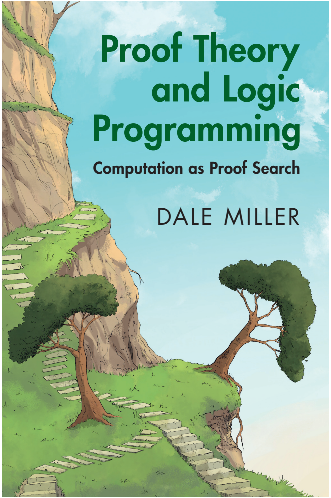
Preprint available from my web page. https://www.lix.polytechnique.fr/Labo/Dale.Miller/ptlp/

Organizes everything I learned about the intersection of proof theory and logic programming during four decades (1985-2025).

Uses classical, intuitionistic, and linear logic (first-order and higher-order) to design and reason about logic programs.



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Outline

The Abella proof assistant

Proof theory basics

Application of proof theory

Future work and conclusion

Abella

An interactive theorem prover well-suited for reasoning about the meta-theory of languages and logics involving binding.

- Various results on the λ-calculus involving big-step evaluation, small-step evaluation, and typing judgments
- Cut-admissibility for a sequent calculus
- Part 1a and Part 2a of the POPLmark challenge
- Several theorems about the π-calculus
- Takahashi's proof of the Church-Rosser theorem
- Tait's logical relations proof of weak normalization for STLC
- Girard's proof of strong normalization of STLC

Abella: A System for Reasoning about Relational Specifications by Baelde, Chaudhuri, Gacek, Miller, Nadathur, Tiu, and Wang. Journal of Formalized Reasoning 7(2), 2014, 1-89.

Abella is based on the sequent calculus

A goal in Abella is a sequent (à la Gentzen) displayed as

```
Variables: x1 ... xm
H1 : A1
...
Hn : An
============================
 C
```

- $x_1, \ldots, x_m$ are universally quantified variables (eigenvariables)
- `H1`, ..., `Hn` are hypothesis labels
- $A_1, \ldots, A_n$ are the hypotheses
- C is the conclusion

This goal denotes the sequent

$$x_1 : \tau_1, \ldots, x_m : \tau_m :: A_1, \ldots, A_n \vdash C.$$

The left-hand side is the context of this goal.

Abella is based on relations, not functions

Abella is based on quantificational logic: it contains

- relations (encoded as predicates) and
- term constructors.

Abella does not have a primitive notion of functions. These must be encoded as relations.
Example: Instead of writing $(a * x) + b \leq c$, write

$$\exists x_1.\exists x_2.\ \mathit{times}\ a\ x\ x_1 \wedge\ \mathit{plus}\ x_1\ b\ x_2 \wedge \mathit{lesseq}\ x_2\ c$$

Example sequents in Abella

Theorem: For all natural numbers d, n, and m, if $d \mid n$ and $d \mid m$ then $d \mid (n + m)$.

```
Theorem div_plus : forall D N M P,
  divides D N -> divides D M -> plus N M P ->
     divides D P.
```

Determinacy lemmas:

```
Theorem times_det : forall A B C D,
   times A B C -> times A B D -> C = D.
Theorem plus_det  : forall A B C D,
   plus  A B C -> plus  A B D -> C = D.
```

Totality lemmas:

```
Theorem times_total : forall A B,
   nat A -> nat B -> exists C, times A B C.
Theorem plus_total  : forall A B,
   nat A -> nat B -> exists C, times A B C.
```

An example goal

```
Variables: D N M P P1 P2 P6 Q XtY XtZ
H3: plus N M P
H4: times D P1 N
H5: times D P2 M
H12: plus P1 P2 Q
H13: times D Q P6
H14: times D P1 XtY
H15: times D P2 XtZ
H16: plus XtY XtZ P6
================================
 exists P1, times D P1 P
```

This goal specifies two ordered hypergraphs where variables denote nodes, e.g., the hypergraph for `times` contains the triples

```
(D,P1,N),(D,P2,M),(D,Q,P6),(D,P1,XtY),(D,P2,XtZ)
```

which, in turn, denote the equations

```
D*P1=N, D*P2=M, D*Q=P6, D*P1=XtY, D*P2=XtZ
```

Forward chaining as hypergraph rewriting

```
Theorem plus_comm  : forall M N K,
  plus M N K -> plus N M K.

Theorem plus_assoc : forall A B C AB ABC,
  plus A B AB -> plus AB C ABC ->
     exists BC, plus B C BC /\ plus A BC ABC.
```

How can such formulas be converted into hypergraph rewriting?

$A \supset B$

$A_1 \supset A_2 \supset (B_1 \wedge B_2) \equiv (A_1 \wedge A_2) \supset (B_1 \wedge B_2)$

The following are less obvious.

$$A_1 \supset A_2 \supset (B_1 \vee B_2) \qquad A_1 \supset A_2 \supset x = y$$

$$A_1 \supset A_2 \supset \exists x.B \qquad A_1 \supset A_2 \supset ff$$

Or should we be using linear logic?

$A_1 \multimap A_2 \multimap B \equiv (A_1 \otimes A_2) \multimap B$

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Elements of the sequent calculus: the micro rules

There are three kinds of (micro) rules:

1. Structural rules: weakening and contraction
2. Identity rules: cut and initial
3. Introduction rules put logical connectives into the concluding sequents.

$$\frac{\Sigma :: \Gamma \vdash B \quad \Sigma :: \Gamma \vdash C}{\Sigma :: \Gamma \vdash B \wedge C} \wedge_r \qquad \frac{\Sigma :: \Gamma, B \vdash C}{\Sigma :: \Gamma \vdash B \supset C} \supset_r$$

$$\frac{\Sigma :: \Gamma \vdash B \quad \Sigma :: \Gamma, C \vdash E}{\Sigma :: \Gamma, B \supset C \vdash E} \supset_l$$

Elements of the sequent calculus: the macro rules

Focused proof systems permit the systematic construction of macro rules, usually called synthetic inference rules.
For example, the formula

$$\forall x \forall y \forall z\,(path(x,y) \supset path(y,z) \supset path(x,z))$$

(which specifies that the predicate *path* is transitive) can be converted into the following synthetic rules:

$$\frac{\Sigma :: \Gamma \vdash path(x,y) \quad \Sigma :: \Gamma \vdash path(y,z)}{\Sigma :: \Gamma \vdash path(x,z)}\ \text{back-chaining}$$

$$\frac{\Sigma :: path(x,y),\ path(y,z),\ path(x,z),\ \Gamma \vdash E}{\Sigma :: path(x,y),\ path(y,z), \Gamma \vdash E}\ \text{forward-chaining}$$

Elements of the sequent calculus: equality

Here, we treat equality as a logical connective: that is, it has a left and right introduction rule.

$$\frac{}{\Sigma :: \Gamma \vdash t = t} =_r$$

$$\frac{}{\Sigma :: s = t, \Gamma \vdash C} =_l \qquad\qquad \frac{\Sigma\theta :: \Gamma\theta \vdash C\theta}{\Sigma :: s = t, \Gamma \vdash C} =_l$$

where t and s not unifiable. where $\theta = \mathrm{mgu}(t, s)$.

If we have no constructors (i.e., terms are just eigenvariables), then these rules simplify to the following.

$$\frac{}{\Sigma :: \Gamma \vdash x = x} =_r \qquad\qquad \frac{\Sigma, x :: \Gamma[x/y] \vdash C[x/y]}{\Sigma, x, y :: x = y, \Gamma \vdash C} =_l$$

Elements of the sequent calculus: linear vs classical zones

Sequents can be split into two zones: one set Γ for classical assumptions, and one multiset Δ for linear logic assumptions.

$$\frac{\Gamma, B; \Delta \vdash C}{\Gamma; \Delta \vdash B \supset C} \qquad \frac{\Gamma; \Delta, B \vdash C}{\Gamma; \Delta \vdash B \multimap C}$$

$$\frac{\Gamma; \Delta_1 \vdash B \quad \Gamma; \Delta_2, C \vdash E}{\Gamma; \Delta_1, \Delta_2, B \multimap C \vdash E} \qquad \frac{\Gamma; \Delta_1 \vdash B \quad \Gamma; \Delta_2 \vdash C}{\Gamma; \Delta_1, \Delta_2 \vdash B \otimes C}$$

Subexponentials and more zones

In linear logic, the ! and ? exponentials are not canonical: that is, linear logic supports

- many subexponentials !, !, !, etc.
- many zones: $\Gamma; \Delta_1; \Delta_2; \Delta_3 \vdash E$

Thus, different multisets (and, by extension, hypergraphs) can be located in different places. [Nigam & M, PPDP, 2009].

Preliminary work on encoding Milner's bigraphs in linear logic uses subexponentials [Chaudhuri & Reis, LPAR, 2015].

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Multiset rewriting as forward chaining

Focusing on formulas yields synthetic rules. For example:

$$\alpha : a \otimes b \multimap c \otimes d \qquad \frac{\Gamma; c, d, \Delta \vdash E}{\Gamma; a, b, \Delta \vdash E} \qquad \frac{\Gamma, a; c, d, \Delta \vdash E}{\Gamma, a; b, \Delta \vdash E}$$

$$\beta : e \multimap f \otimes g \qquad \frac{\Gamma; f, g, \Delta \vdash E}{\Gamma; e, \Delta \vdash E} \qquad \frac{\Gamma, e; f, g, \Delta \vdash E}{\Gamma, e; \Delta \vdash E}$$

We can do these actions in either order: $\alpha; \beta$ and $\beta; \alpha$ yield the same changes in sequents. <u>Multifocusing</u> also provides for $\alpha|\beta$.

Consider: $\gamma : a \otimes \mathit{lock} \multimap c \otimes \mathit{lock} \qquad \delta : b \otimes \mathit{lock} \multimap d \otimes \mathit{lock}$

Both $\gamma; \delta$ and $\delta; \gamma$ are possible but not their parallel composition, unless there is more than one occurrence of *lock* in Δ or $\mathit{lock} \in \Gamma$ (in which case, you don't really have a lock).

Naming and sharing

Administrative normal form (ANF) is a well known technique from compiling: apply functions only to named values and return a named value.

E.g., instead of $(a * x) + b \leq c$, write

```
name x1 = (a  * x) in
name x2 = (x1 + b) in
lesseq x2 c
```

This expression can be translated to the simple formula:

$$\exists x_1.\ \mathit{times}\ a\ x\ x_1 \wedge \exists x_2.\ \mathit{plus}\ x_1\ b\ x_2 \wedge \mathit{lesseq}\ x_2\ c$$

Naming and sharing (continued)

The use of names in this setting allows for sharing of common subexpressions.

```
name x1 = (f a a) in
name x2 = (f x1 x1) in
name x3 = (f x2 x2) in
name x4 = (f x3 x3) in
name x5 = (f x4 x4) in
x5;
```

If we remove the sharing, we get a big term containing, for example, 16 copies of the subexpression `(f a a)`.

If proofs are terms and formulas are types, then forward chaining in such proofs yield terms in ANF. See: [M & Wu, CSL 2023].

Forward chaining saturation

Geometric formulas: $\forall x_1 \ldots \forall x_m.\ A_1 \wedge \cdots \wedge A_n \supset G,$ where

- $A_1, \ldots, A_n$ are atomic
- G is built from atoms, $=$, $f\!f$, $t\!t$, $\wedge$, $\vee$, $\exists$.

Focused proof theory provides a notion of productive proofs.

Saturation occurs when a proof becomes unproductive.

Various syntactic characterizations can guarantee safe for saturation.

Using ANF, determinacy axioms, and saturation provides a proof theory (and proof search) account for congruence closure.

See: [Chaudhuri, Gantait, & M, Tableaux 2025]

Outline

Future work: Bindings

Hypergraphs and term graphs often encode expressions, and these expressions often contain binders.

The POPLmark challenge has revealed that the theorem proving community has not developed a canonical approach to encoding such binders: many proposals exist (de Bruijn numerals, nominal constants, HOAS, locally nameless, etc).

The term graph literature contains some approaches to capturing bindings: [Schneider, Kœhler, & Steuwer, PLDI 2025], [Tiurin, Ghica, & Hu, 2025], [Yasen & Ueda, 2017, 2018]

Future work: Bindings (continued)

The sequent calculus provides a natural, immediate, and powerful approach to binders, based on the notion of binder mobility.

> *There is no such thing as a free variable.*
> *(Perlis's epigram #47)*

The initial motivation for building the Abella prover was to capture this notion of binder mobility and the associated notion of λ-tree syntax and ∇-quantification.

Current situation: the notion of equality on λ-graph structures [Condoluci, Accattoli, Sacerdoti Coen, PPDP 2019] has a direct specification in Abella.

Future: Consider extending this work to Term Graphs more generally and compare with existing proposals.

Conclusions

- Proof-theoretic foundations has some overlaps with hypergraphs and term graphs.
- The implementations of hypergraph rewriting, e-graphs, etc, should find a home in Abella.
- I hope to force the proof theory approach to bindings to provide another approach to bindings in hypergraphs.

Thanks

Questions?

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