REDUCIBLE GRAPHS AND BISIMILARITY OF 1-FREE STAR EXPRESSIONS

Bas Laarakker & Tobias Kappé, Leiden University

TERMGRAPH

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- With binary Kleene star $e^{\circledast} f$, 1 is *not* definable.
- Immediate termination is not possible in 1-free expressions.

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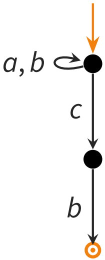

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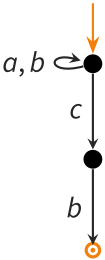

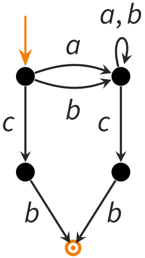

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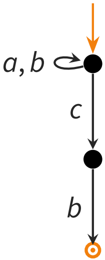

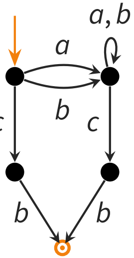

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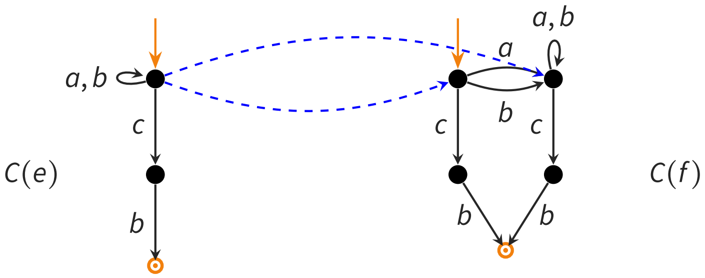

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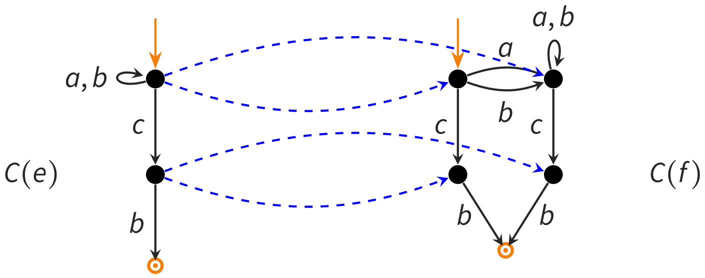

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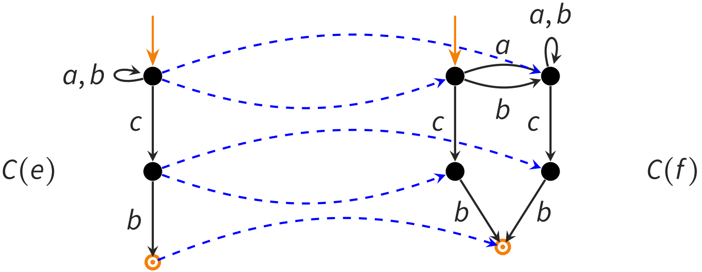

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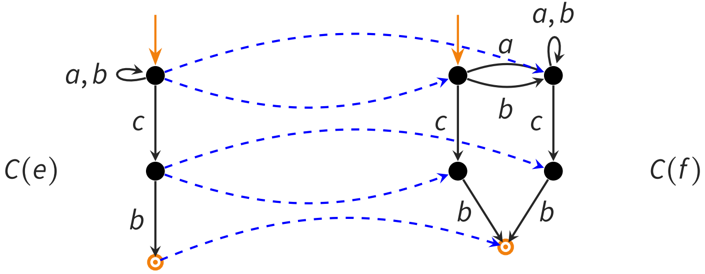

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The two charts are bisimilar: $C(e) \leftrightarroweq C(f)$.

We say that the expressions e and f are bisimilar when $C(e) \leftrightarroweq C(f)$.

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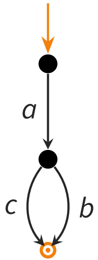
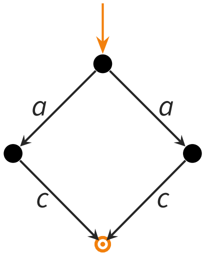
The converse is not true:



$a \cdot (b + c)$



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PROOF SYSTEM BBP FOR BISIMILARITY

Axioms for language equivalence of 1-free expressions:

(B1)	$x + y = y + x$	(B2)	$(x + y) + z = x + (y + z)$
(B3)	$x + x = x$	(B4)	$x + 0 = x$
(B5)	$(x + y) \cdot z = x \cdot z + y \cdot z$	(L1)	$x \cdot (y + z) = x \cdot y + x \cdot z$
(B6)	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	(B7)	$0 \cdot x = 0$
(L2)	$x \cdot 0 = 0$	(BKS1)	$x \cdot (x^{\circledast} y) + y = x^{\circledast} y$
(BKS2)	$(x^{\circledast} y) \cdot z = x^{\circledast}(y \cdot z)$	(RSP)	$\dfrac{x = (y \cdot x) + z}{x = y^{\circledast} z}$

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- (Grabmayer & Fokkink 2020): The axioms are complete for bisimilarity

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Definition

A *(provable) solution* to a chart C, with set of nodes V, is a function $s : V \to \mathrm{StExp}(A)$ such that the following holds for all $v \in V$:

$$s(v) =_{\mathrm{BBP}} \sum_{v \xrightarrow{a} \surd} a + \sum_{v \xrightarrow{b} w} b \cdot s(w)$$

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3. The class $\mathcal{C}$ is closed under homomorphic images (quotients).

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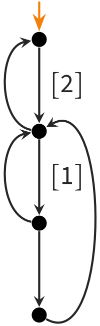

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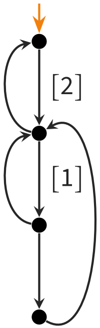

Challenge: The class of LLEE charts is closed under homomorphic images.

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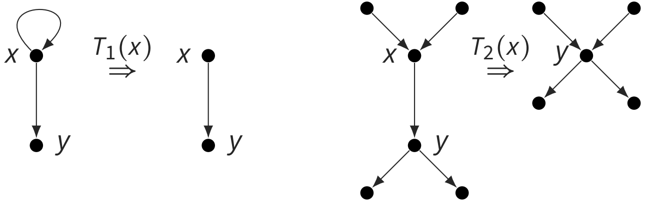

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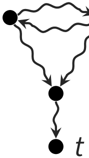

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"Forbidden subgraph"

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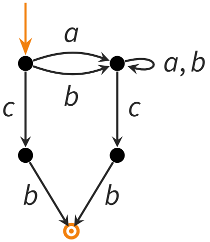

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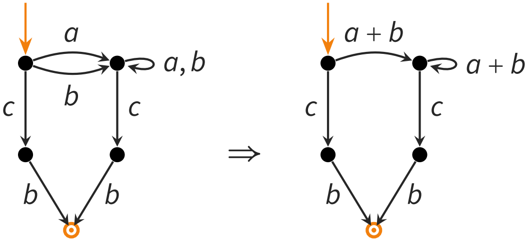

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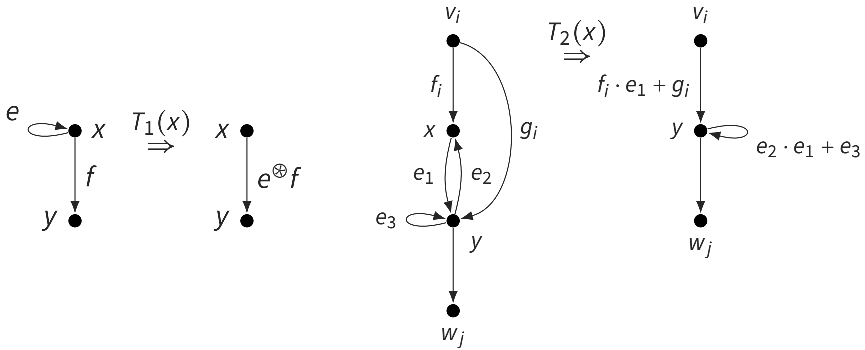

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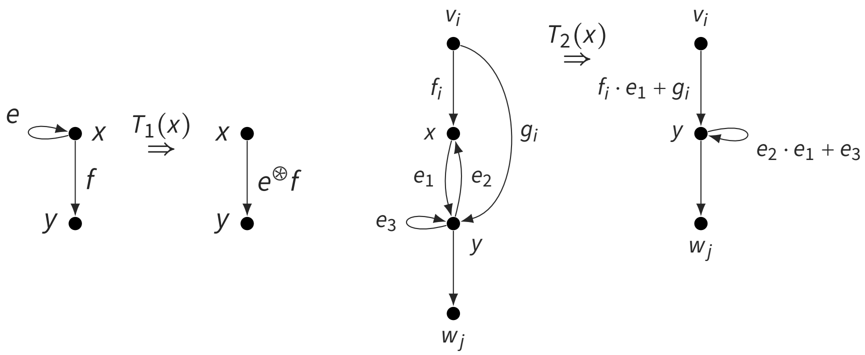

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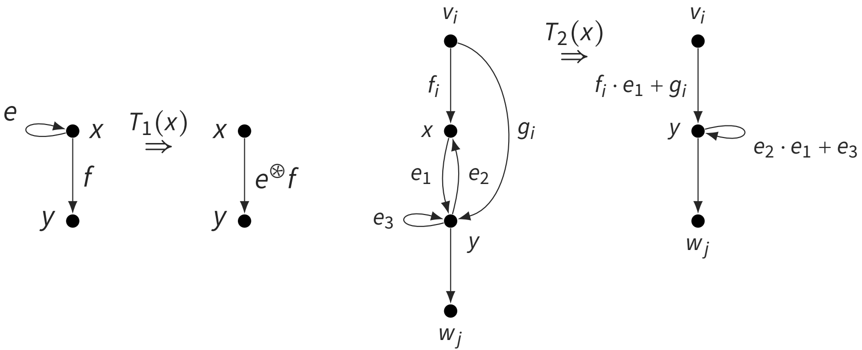

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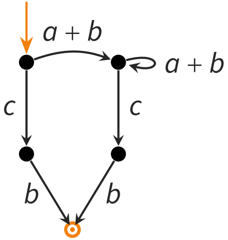

$\longrightarrow^2$

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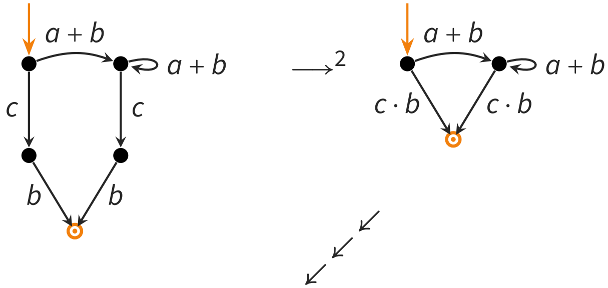

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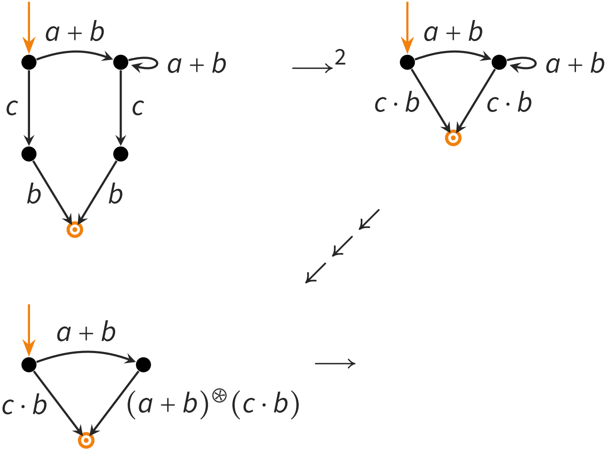

$a + b$
$a + b$
c
c
b
b
$\longrightarrow^2$
$a + b$
$a + b$
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$(a + b)^{\circledast}(c \cdot b)$
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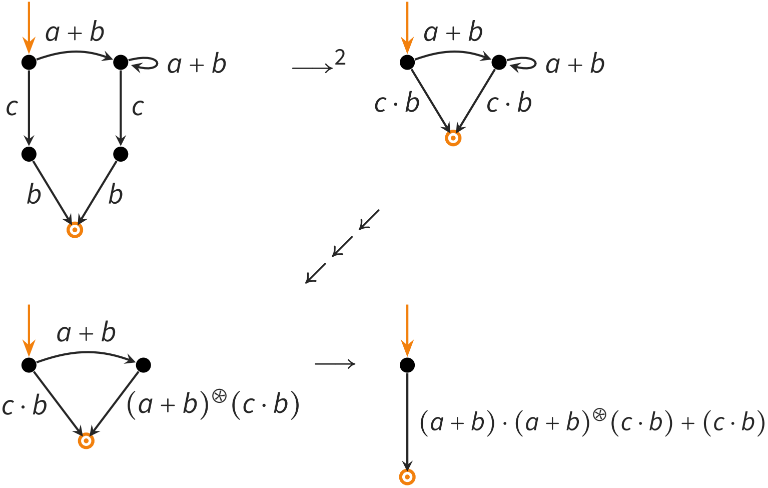
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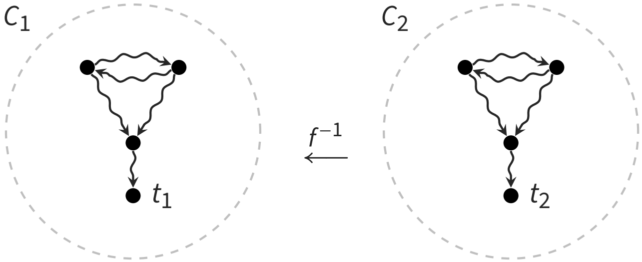

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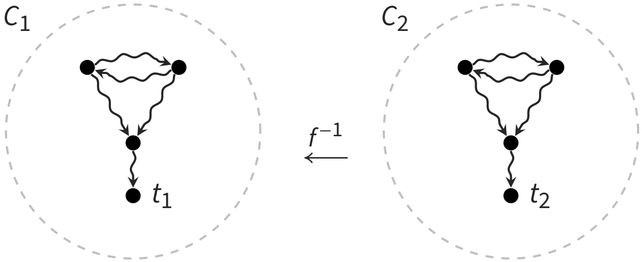

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A completeness proof of BBP restricted to strongly normed expressions.

FUTURE WORK

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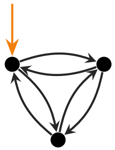

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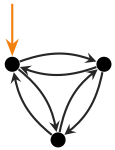

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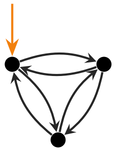

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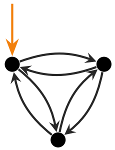

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Thank you!